

The smallest degree sum that yields potentially $K_{r+1} - Z$ -graphical Sequences *

Chunhui Lai

Department of Mathematics, Zhangzhou Teachers College,
Zhangzhou, Fujian 363000, P. R. of CHINA.
e-mail: zjlaichu@public.zzptt.fj.cn

Abstract

Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). We use the symbol Z_4 to denote $K_4 - P_2$. A sequence S is potentially $K_m - H$ -graphical if it has a realization containing a $K_m - H$ as a subgraph. Let $\sigma(K_m - H, n)$ denote the smallest degree sum such that every n -term graphical sequence S with $\sigma(S) \geq \sigma(K_m - H, n)$ is potentially $K_m - H$ -graphical. In this paper, we determine the values of $\sigma(K_{r+1} - Z, n)$ for $n \geq 5r + 19, r + 1 \geq k \geq 5, j \geq 5$ where Z is a graph on k vertices and j edges which contains a graph Z_4 but not contains a cycle on 4 vertices. We also determine the values of $\sigma(K_{r+1} - Z_4, n)$, $\sigma(K_{r+1} - (K_4 - e), n)$, $\sigma(K_{r+1} - K_4, n)$ for $n \geq 5r + 16, r \geq 4$.

Key words: subgraph; degree sequence; potentially $K_{r+1} - Z$ -graphic; potentially $K_{r+1} - Z_4$ -graphic sequence

AMS Subject Classifications: 05C07, 05C35

1 Introduction

The set of all non-increasing nonnegative integers sequence $\pi = (d_1, d_2, \dots, d_n)$ is denoted by NS_n . A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is

*Project Supported by NNSF of China(10271105), NSF of Fujian(Z0511034), Fujian Provincial Training Foundation for "Bai-Quan-Wan Talents Engineering", Project of Fujian Education Department and Project of Zhangzhou Teachers College.

called a realization of π . The set of all graphic sequences in NS_n is denoted by GS_n . A graphical sequence π is potentially H -graphical if there is a realization of π containing H as a subgraph, while π is forcibly H -graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r + 1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1} -graphic. Let $\sigma(\pi) = d_1 + d_2 + \dots + d_n$, and $[x]$ denote the largest integer less than or equal to x . If G and G_1 are graphs, then $G \cup G_1$ is the disjoint union of G and G_1 . If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. We denote $G + H$ as the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. Let K_k , C_k , T_k , and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k + 1$ vertices, and a path on $k + 1$ vertices, respectively. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). We use the symbol Z_4 to denote $K_4 - P_2$. We use the symbol $G[v_1, v_2, \dots, v_k]$ to denote the subgraph of G induced by vertex set $\{v_1, v_2, \dots, v_k\}$. We use the symbol $\epsilon(G)$ to denote the numbers of edges in graph G .

Given a graph H , what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdős [2] in 1938 and in general by Turán [19]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n -term graphical sequence π with $\sigma(\pi) \geq l$ is forcibly H -graphical. Here we consider the following variant: determine the minimum even integer l such that every n -term graphical sequence π with $\sigma(\pi) \geq l$ is potentially H -graphical. We denote this minimum l by $\sigma(H, n)$. Erdős, Jacobson and Lehel [4] showed that $\sigma(K_k, n) \geq (k - 2)(2n - k + 1) + 2$ and conjectured that equality holds. They proved that if π does not contain zero terms, this conjecture is true for $k = 3$, $n \geq 6$. The conjecture is confirmed in [5],[14],[15],[16] and [17].

Gould, Jacobson and Lehel [5] also proved that $\sigma(pK_2, n) = (p - 1)(2n - 2) + 2$ for $p \geq 2$; $\sigma(C_4, n) = 2\lceil \frac{3n-1}{2} \rceil$ for $n \geq 4$. They also pointed out that it would be nice to see where in the range for $3n - 2$ to $4n - 4$, the value $\sigma(K_4 - e, n)$ lies. Luo [18] characterized the potentially C_k graphic sequence for $k = 3, 4, 5$. Lai [7] determined $\sigma(K_4 - e, n)$ for $n \geq 4$. Yin, Li and Mao[21] determined $\sigma(K_{r+1} - e, n)$ for $r \geq 3$, $r + 1 \leq n \leq 2r$ and $\sigma(K_5 - e, n)$ for $n \geq 5$. Yin and Li [20] gave a good method (Yin-Li method) of determining the values $\sigma(K_{r+1} - e, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$ (In fact, Yin and Li[20] also determining the values $\sigma(K_{r+1} - ke, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$). After reading[20], using Yin-Li method Yin [22] determined $\sigma(K_{r+1} - K_3, n)$ for $n \geq 3r + 5$, $r \geq 3$. Lai [8] determined $\sigma(K_5 - K_3, n)$, for $n \geq 5$. Lai [9] gave a lower bound of $\sigma(K_{t+p} - K_p, n)$. Lai [10,11] determined $\sigma(K_5 - C_4, n)$, $\sigma(K_5 - P_3, n)$ and $\sigma(K_5 - P_4, n)$, for

$n \geq 5$. Determining $\sigma(K_{r+1} - H, n)$, where H is a tree on 4 vertices is more useful than a cycle on 4 vertices (for example, $C_4 \not\subset C_i$, but $P_3 \subset C_i$ for $i \geq 5$). So, after reading [20] and [22], using Yin-Li method Lai and Hu [12] determined $\sigma(K_{r+1} - H, n)$ for $n \geq 4r + 10, r \geq 3, r + 1 \geq k \geq 4$ and H be a graph on k vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices and $\sigma(K_{r+1} - P_2, n)$ for $n \geq 4r + 8, r \geq 3$. Using Yin-Li method Lai and Sun [13] determined $\sigma(K_{r+1} - (kP_2 \cup tK_2), n)$ for $n \geq 4r + 10, r + 1 \geq 3k + 2t, k + t \geq 2, k \geq 1, t \geq 0$. To now, the problem of determining $\sigma(K_{r+1} - H, n)$ for H not containing a cycle on 3 vertices and sufficiently large n has been solved. In this paper, using Yin-Li method we prove the following two theorems.

Theorem 1.1. If $r \geq 4$ and $n \geq 5r + 16$, then

$$\sigma(K_{r+1} - K_4, n) = \sigma(K_{r+1} - (K_4 - e), n) = \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

Theorem 1.2. If $n \geq 5r + 19, r + 1 \geq k \geq 5$, and $j \geq 5$, then

$$\sigma(K_{r+1} - Z, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

where Z is a graph on k vertices and j edges which contains a graph Z_4 but not contains a cycle on 4 vertices.

There are a number of graphs on k vertices and j edges which contains a graph Z_4 but not contains a cycle on 4 vertices.

2 Preparations

In order to prove our main result, we need the following notations and results.

Let $\pi = (d_1, \dots, d_n) \in NS_n, 1 \leq k \leq n$. Let

$$\pi_k'' = \begin{cases} (d_1 - 1, \dots, d_{k-1} - 1, d_{k+1} - 1, \dots, d_{d_k+1} - 1, d_{d_k+2}, \dots, d_n), \\ \text{if } d_k \geq k, \\ (d_1 - 1, \dots, d_{d_k} - 1, d_{d_k+1}, \dots, d_{k-1}, d_{k+1}, \dots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Denote $\pi'_k = (d'_1, d'_2, \dots, d'_{n-1})$, where $d'_1 \geq d'_2 \geq \dots \geq d'_{n-1}$ is a rearrangement of the $n-1$ terms of π''_k . Then π'_k is called the residual sequence obtained by laying off d_k from π .

Theorem 2.1[20] Let $n \geq r+1$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r+1} \geq r$. If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-1$, then π is potentially A_{r+1} -graphic.

Theorem 2.2[20] Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r+1} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially A_{r+1} -graphic.

Theorem 2.3[20] Let $n \geq r+1$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r+1} \geq r-1$. If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-1$, then π is potentially $K_{r+1}-e$ -graphic.

Theorem 2.4[20] Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-1} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1}-e$ -graphic.

Theorem 2.5[6] Let $\pi = (d_1, \dots, d_n) \in NS_n$ and $1 \leq k \leq n$. Then $\pi \in GS_n$ if and only if $\pi'_k \in GS_{n-1}$.

Theorem 2.6[3] Let $\pi = (d_1, \dots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for any $t, 1 \leq t \leq n-1$,

$$\sum_{i=1}^t d_i \leq t(t-1) + \sum_{j=t+1}^n \min\{t, d_j\}.$$

Theorem 2.7[5] If $\pi = (d_1, d_2, \dots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π .

Theorem 2.8[9] If $n \geq p+t$, then $\sigma(K_{p+t} - K_p, n) \geq 2[(p+2t-3)n + p + 2t + 1 - pt - t^2]/2$.

Lemma 2.1 [22] If $\pi = (d_1, d_2, \dots, d_n) \in NS_n$ is potentially $K_{r+1}-e$ -graphic, then there is a realization G of π containing $K_{r+1}-e$ with the $r+1$ vertices v_1, \dots, v_{r+1} such that $d_G(v_i) = d_i$ for $i = 1, 2, \dots, r+1$ and $e = v_r v_{r+1}$.

Lemma 2.2 [12] Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-2} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1}-P_2$ -graphic.

Lemma 2.3 Let $\pi = (d_1, \dots, d_n) \in GS_n$ and G be a realization of π . If $\epsilon(G[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 1$, then there is a realization H of π such that $d_H(v_i) = d_i$ for $i = 1, 2, \dots, r+1$ and $v_r v_{r+1} \notin E(H)$.

The proof is similar to the proof of Lemma 2.1.

3 Proof of Main results.

Lemma 3.1. Let $n \geq 2r$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-1} \geq r$, $d_{r+1} \geq r-1$. If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-2$, then π is potentially $K_{r+1}-e$ -graphic.

Proof. We consider the following two cases.

Case 1: $d_{r+1} \geq r$.

If $d_{r-1} \geq r+1$.

Then π is potentially $K_{r+1} - e$ -graphic by Theorem 2.3.

If $d_{r-1} = r$, then $d_{r-1} = d_r = d_{r+1} = r$

Suppose π is not potentially $K_{r+1} - e$ -graphic. Let H be a realization of π , then $\epsilon(H[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 2$. Let $S = (d_1, d_2, \dots, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, \dots, d_n)$, then by Theorem 2.1, S is potentially A_{r+1} -graphic (Denote $S' = (d'_1, d'_2, \dots, d'_n)$, where $d'_1 \geq d'_2 \geq \dots \geq d'_n$ is a rearrangement of the n terms of S . Therefore $S' \in GS_n$ by Lemma 2.3. Then S' satisfies the conditions of Theorem 2.1). Therefore, there is a realization G of S with v_1, v_2, \dots, v_{r+1} ($d(v_i) = d_i, i = 1, 2, \dots, r-1$, $d(v_r) = d_r + 1, d(v_{r+1}) = d_{r+1} + 1$), the $r+1$ vertices of highest degree containing a K_{r+1} . Hence, $G - v_{r+1}v_r$ is a realization of π . Thus, π is potentially $K_{r+1} - e$ -graphic, which is a contradiction.

Case 2: $d_{r+1} = r-1$, then the residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off $d_{r+1} = r-1$ from π satisfies: $d'_1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-1} = d'_{r-2} \geq 2(r-1) - (r-2), d'_{(r-1)+1} = d'_r \geq r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$ -graphic. Therefore, π is potentially $K_{r+1} - e$ -graphic by $\{d_1 - 1, \dots, d_{r-1} - 1\} \subseteq \{d'_1, \dots, d'_r\}$ and Theorem 2.7.

Lemma 3.2. Let $n \geq 2r$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-2} \geq r+1, d_{r+1} \geq r, d_r - 1 \geq d_{d_{r+1}+2}$. If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$, then π is potentially A_{r+1} -graphic.

Proof. The residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq 2(r-1) - (r-3), d'_{(r-1)-1} = d'_{r-2} \geq 2(r-1) - (r-2), d'_{(r-1)+1} = d'_r \geq r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$ -graphic. Therefore, π is potentially A_{r+1} -graphic by $\{d_1 - 1, \dots, d_{r-1} - 1\} = \{d'_1, \dots, d'_r\}$ and Theorem 2.7.

Lemma 3.3 Let $n \geq 2r+2, r \geq 4$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-2} \geq r-1$ and $d_{r+1} \geq r-2$,

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$, then π is potentially $K_{r+1} - Z_4$ -graphic.

Proof. We consider the following two cases.

Case 1: $d_{r+1} \geq r-1$.

Subcase 1.1: $d_{r-1} \geq r+1$.

If $d_{r-2} \geq r+2$, then π is potentially $K_{r+1} - e$ -graphic by Theorem 2.3.

Hence, π is potentially $K_{r+1} - Z_4$ -graphic.

If $d_{r-2} = r + 1$, then $d_{r-3} - 1 \geq d_{r-2}$. The residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq 2(r-1) - (r-3)$, $d'_{(r-1)-1} = d'_{r-2} \geq r-1$, $d'_{(r-1)+1} = d'_r \geq (r-1) - 1$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - Z_4$ -graphic by $\{d_1 - 1, \dots, d_{r-3} - 1\} \subseteq \{d'_1, \dots, d'_r\}$ and Lemma 2.1.

Subcase 1.2: $d_{r-1} \leq r$. then $d_{r-3} - 1 \geq d_{r-1}$. The residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq 2(r-1) - (r-3)$, $d'_{(r-1)-1} = d'_{r-2} \geq r-1$, $d'_{(r-1)+1} = d'_r \geq (r-1) - 1$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - Z_4$ -graphic by $\{d_1 - 1, \dots, d_{r-3} - 1\} \subseteq \{d'_1, \dots, d'_r\}$ and Lemma 2.1.

Case 2: $d_{r+1} = r - 2$.

If $d_{r-1} < d_{r-2}$.

If $d_{r-2} \geq r$, then the residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off $d_{r+1} = r - 2$ from π satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \dots, r - 2$, (2) $d'_1 = d_1 - 1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$, and $d'_{(r-1)+1} = d'_r = d_r \geq r - 2$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - Z_4$ -graphic by $\{d_1 - 1, \dots, d_{r-2} - 1, d_{r-1}, d_r\} = \{d'_1, \dots, d'_r\}$ and Lemma 2.1.

If $d_{r-2} = r - 1$, then $d_{r-1} = d_r = r - 2$ and

$$\begin{aligned} \sigma(\pi) &\leq (r-3)(n-1) + r-1 + (r-2)(n-r+2) \\ &= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+2) \\ &= (r-1)(2n-r) - 3(n-r) - 2 \end{aligned}$$

Hence, $\pi = ((n-1)^{r-3}, (r-1)^1, (r-2)^{n-r+2})$ and $n-r$ is even. Clearly, π is potentially $K_{r+1} - Z_4$ -graphic.

If $d_{r-1} = d_{r-2}$ and $d_{r-3} \geq d_r$, then π'_{r+1} satisfies: $d'_1 \geq d_1 - 1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$ and $d'_{(r-1)+1} = d'_r \geq r - 2$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - Z_4$ -graphic by $\{d_{r-1}, d_r, d_1 - 1, \dots, d_{r-2} - 1\} = \{d'_1, \dots, d'_r\}$ and Lemma 2.1.

If $d_{r-1} = d_{r-2}$ and $d_{r-3} = d_r$, then $d_{r-3} = d_{r-2} = d_{r-1} = d_r \geq r + 3$. Let H be a realization of π . Since $d_{r+1} = r - 2$, then there is $i, j \leq r$ such that $v_{r+1}v_i, v_{r+1}v_j \notin E(H)$. Let $S = (d_1, d_2, \dots, d_i + 1, \dots, d_j + 1, \dots, d_r, d_{r+1} + 2, \dots, d_n)$, then by Theorem 2.1, S is potentially A_{r+1} -graphic (Denote $S' = (d'_1, d'_2, \dots, d'_n)$, where $d'_1 \geq d'_2 \geq \dots \geq d'_n$ is a rearrangement of the n terms of S . Therefore $S' \in GS_n$. Then S' satisfies the conditions of Theorem 2.1). Therefore, there is a realization G of S with v_1, v_2, \dots, v_{r+1} ($d(v_t) = d_t, t \neq i, j, r+1, d(v_i) = d_i + 1, d(v_j) = d_j + 1, d(v_{r+1}) = d_{r+1} + 2$), the $r+1$ vertices of highest degree containing a K_{r+1} . Hence, $G -$

$\{v_{r+1}v_i, v_{r+1}v_j\}$ is a realization of π . Thus, π is potentially $K_{r+1} - Z_4$ -graphic.

Lemma 3.4 Let $n \geq 2r + 2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-t} \geq r$. If $d_{2r+2} \geq r - 1$, then π is potentially $K_{r+1} - K_{1,t}$ -graphic.

Proof. We consider the following two cases.

Case 1: If $d_{r-1} \geq r$. Then π is potentially $K_{r+1} - e$ -graphic by Theorem 2.4. Hence, π is potentially $K_{r+1} - K_{1,t}$ -graphic.

Case 2: $d_{r-1} \leq r - 1$, that is, $d_{r-1} = r - 1$, then $d_{r-1} = d_r = d_{r+1} = \dots = d_{2r+2} = r - 1$ and π'_{r+1} satisfies: $d'_{(r-1)+1} = d'_r \geq r - 1$ and $d'_{2(r-1)+2} = d'_{2r} \geq (r - 1) - 1$. By Theorem 2.2, π'_{r+1} is potentially A_r -graphic. Therefore, π is potentially $K_{r+1} - K_{1,t}$ -graphic by $\{d_1 - 1, \dots, d_{r-t} - 1\} \subseteq \{d'_1, \dots, d'_r\}$ and Theorem 2.7.

Lemma 3.5 Let $n \geq 2r + 2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-4} \geq r$,

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

If $d_{2r+2} \geq r - 1$, then π is potentially $K_{r+1} - (P_2 \cup K_2)$ -graphic.

Proof. We consider the following two cases.

Case 1: If $d_{r-2} \geq r$. Then π is potentially $K_{r+1} - P_2$ -graphic by Lemma 2.2. Hence, π is potentially $K_{r+1} - (P_2 \cup K_2)$ -graphic.

Case 2: $d_{r-2} = r - 1$.

Subcase 2.1: $d_{r-3} \geq r$, then $d_{r-3} \geq d_r + 1 = d_{r+1} + 1 = r > r - 1 = d_{r-2} = d_{r-1}$. Suppose π is not potentially $K_{r+1} - (P_2 \cup K_2)$ -graphic. Let H be a realization of π , then $\epsilon(H[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$. Let $S = (d_1, d_2, \dots, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, \dots, d_n)$, then by Theorem 2.4, S is potentially $K_{r+1} - e$ -graphic (Denote $S' = (d'_1, d'_2, \dots, d'_n)$, where $d'_1 \geq d'_2 \geq \dots \geq d'_n$ is a rearrangement of the n terms of S . Therefore $S' \in GS_n$ by Lemma 2.3. Then S' satisfies the conditions of Theorem 2.4). Therefore, there is a realization G of S with v_1, v_2, \dots, v_{r+1} ($d(v_i) = d_i, i = 1, 2, \dots, r - 1, d(v_r) = d_r + 1, d(v_{r+1}) = d_{r+1} + 1$), the $r + 1$ vertices of highest degree containing a $K_{r+1} - e$ and $e = v_{r-1}v_{r-2}$ by Lemma 2.1. Hence, $G - v_{r+1}v_r$ is a realization of π . Thus, π is potentially $K_{r+1} - (P_2 \cup K_2)$ -graphic, which is a contradiction.

Subcase 2.2: $d_{r-3} = r - 1$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-4)(n-1) + (r-1)(n-r+4) \\ &= (r-1)(n-1) - 3(n-1) + (r-1)(n-r+1) + 3(r-1) \\ &= (r-1)(2n-r) - 3(n-r) \end{aligned}$$

Since,

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

Hence, π is one of the following: $((n-1)^{r-5}, (n-2)^1, (r-1)^{n-r+4})$, $((n-1)^{r-4}, (r-1)^{n-r+3}, (r-2)^1)$, for $n-r$ is odd, π is one of the following: $((n-1)^{r-4}, (r-1)^{n-r+4})$, $((n-1)^{r-6}, (n-2)^2, (r-1)^{n-r+4})$, $((n-1)^{r-5}, (n-3)^1, (r-1)^{n-r+4})$, $((n-1)^{r-5}, (n-2)^1, (r-1)^{n-r+3}, (r-2)^1)$, $((n-1)^{r-4}, (r-1)^{n-r+3}, (r-3)^1)$, $((n-1)^{r-4}, (r-1)^{n-r+2}, (r-2)^2)$, for $n-r$ is even. Clearly, π is potentially $K_{r+1} - (P_2 \cup K_2)$ -graphic.

Lemma 3.6. If $r \geq 4$ and $n \geq r+1$, then

$$\sigma(K_{r+1} - Z_4, n) \geq \sigma(K_{r+1} - K_4, n).$$

and

$$\sigma(K_{r+1} - K_4, n) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

Proof. Obviously, for $r \geq 4$ and $n \geq r+1$, $\sigma(K_{r+1} - Z_4, n) \geq \sigma(K_{r+1} - K_4, n)$. By Theorem 2.8, for $r \geq 4$ and $n \geq r+1$, $\sigma(K_{r+1} - K_4, n) = \sigma(K_{4+(r-3)} - K_4, n) \geq 2[(4+2(r-3)-3)n+4+2(r-3)+1-4(r-3)-(r-3)^2]/2]$. Hence,

$$\sigma(K_{r+1} - K_4, n) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

Lemma 3.7. If $n \geq r+1, r+1 \geq k \geq 4$, then

$$\sigma(K_{r+1} - H, n) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

where H is a graph on k vertices which not contains a cycle on 4 vertices.

Proof. Let

$$G = \begin{cases} K_{r-3} + (\frac{n-r+1}{2} + 1)K_2, \\ \text{if } n-r \text{ is odd} \\ K_{r-3} + (\frac{n-r+2}{2})K_2 \cup K_1, \\ \text{if } n-r \text{ is even} \end{cases}$$

Then G is a unique realization of

$$\pi = \begin{cases} ((n-1)^{r-3}, (r-2)^{n-r+3}), & \text{if } n-r \text{ is odd} \\ ((n-1)^{r-3}, (r-2)^{n-r+2}, (r-3)^1), & \text{if } n-r \text{ is even} \end{cases}$$

and G clearly does not contain $K_{r+1} - H$, where the symbol x^y means x repeats y times in the sequence. Thus $\sigma(K_{r+1} - H, n) \geq \sigma(\pi) + 2$. Therefore,

$$\sigma(K_{r+1} - H, n) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

The Proof of Theorem 1.1 According to Lemma 3.6 and $\sigma(K_{r+1} - K_4, n) \leq \sigma(K_{r+1} - (K_4 - e), n) \leq \sigma(K_{r+1} - Z_4, n)$, it is enough to verify that for $n \geq 5r + 16$,

$$\sigma(K_{r+1} - Z_4, n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, & \text{if } n-r \text{ is even} \end{cases}$$

We now prove that if $n \geq 5r + 16$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, & \text{if } n-r \text{ is even} \end{cases}$$

then π is potentially $K_{r+1} - Z_4$ -graphic.

If $d_{r-3} \leq r-1$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-4)(n-1) + (r-1)(n-r+4) \\ &= (r-1)(n-1) - 3(n-1) + (r-1)(n-r+4) \\ &= (r-1)(2n-r) - 3(n-r) \\ &< (r-1)(2n-r) - 3(n-r) + 1, \end{aligned}$$

which is a contradiction. Thus, $d_{r-3} \geq r$.

If $d_{r-2} \leq r-2$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-3)(n-1) + (r-2)(n-r+3) \\ &= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+3) \\ &= (r-1)(2n-r) - 3(n-r) - 3 \\ &< (r-1)(2n-r) - 3(n-r) + 1, \end{aligned}$$

which is a contradiction. Thus, $d_{r-2} \geq r-1$.

If $d_{r+1} \leq r-3$, then

$$\begin{aligned}
\sigma(\pi) &= \sum_{i=1}^r d_i + \sum_{i=r+1}^n d_i \\
&\leq (r-1)r + \sum_{i=r+1}^n \min\{r, d_i\} + \sum_{i=r+1}^n d_i \\
&= (r-1)r + 2 \sum_{i=r+1}^n d_i \\
&\leq (r-1)r + 2(n-r)(r-3) \\
&= (r-1)(2n-r) - 4(n-r) \\
&< (r-1)(2n-r) - 3(n-r) + 1,
\end{aligned}$$

which is a contradiction. Thus, $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - Z_4$ -graphic by Lemma 3.3 or Lemma 3.4. If $d_{2r+2} \leq r-2$ and there exists an integer i , $1 \leq i \leq r-3$ such that $d_i \leq 2r-i-1$, then

$$\begin{aligned}
\sigma(\pi) &\leq (i-1)(n-1) + (2r+1-i+1)(2r-i-1) \\
&\quad + (r-2)(n+1-2r-2) \\
&= i^2 + i(n-4r-2) - (n-1) \\
&\quad + (2r-1)(2r+2) + (r-2)(n-2r-1).
\end{aligned}$$

Since $n \geq 5r+16$, it is easy to see that $i^2 + i(n-4r-2)$, consider as a function of i , attains its maximum value when $i = r-3$. Therefore,

$$\begin{aligned}
\sigma(\pi) &\leq (r-3)^2 + (n-4r-2)(r-3) - (n-1) \\
&\quad + (2r-1)(2r+2) + (r-2)(n-2r-1) \\
&= (r-1)(2n-r) - 3(n-r) - n + 5r + 16 \\
&< \sigma(\pi),
\end{aligned}$$

which is a contradiction.

Thus,

$$\sigma(K_{r+1} - Z_4, n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, & \text{if } n-r \text{ is even} \end{cases}$$

for $n \geq 5r+16$.

The Proof of Theorem 1.2 According to Lemma 3.7, it is enough to verify that for $n \geq 5r+19$,

$$\sigma(K_{r+1} - Z, n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

We now prove that if $n \geq 5r + 19$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

then π is potentially $K_{r+1} - Z$ -graphic.

If $d_{r-4} \leq r-1$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-5)(n-1) + (r-1)(n-r+5) \\ &= (r-1)(n-1) - 4(n-1) + (r-1)(n-r+5) \\ &= (r-1)(2n-r) - 4(n-r) \\ &< (r-1)(2n-r) - 3(n-r) - 2, \end{aligned}$$

which is a contradiction. Thus, $d_{r-4} \geq r$.

If $d_{r-2} \leq r-2$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-3)(n-1) + (r-2)(n-r+3) \\ &= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+3) \\ &= (r-1)(2n-r) - 3(n-r) - 3 \\ &< (r-1)(2n-r) - 3(n-r) - 2, \end{aligned}$$

which is a contradiction. Thus, $d_{r-2} \geq r-1$.

If $d_{r+1} \leq r-3$, then

$$\begin{aligned} \sigma(\pi) &= \sum_{i=1}^r d_i + \sum_{i=r+1}^n d_i \\ &\leq (r-1)r + \sum_{i=r+1}^n \min\{r, d_i\} + \sum_{i=r+1}^n d_i \\ &= (r-1)r + 2 \sum_{i=r+1}^n d_i \\ &\leq (r-1)r + 2(n-r)(r-3) \\ &= (r-1)(2n-r) - 4(n-r) \\ &< (r-1)(2n-r) - 3(n-r) - 2, \end{aligned}$$

which is a contradiction. Thus, $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - Z$ -graphic by Lemma 3.3 or Lemma 3.5. If $d_{2r+2} \leq r-2$ and there exists an integer i , $1 \leq i \leq r-3$ such that $d_i \leq 2r-i-1$, then

$$\begin{aligned} \sigma(\pi) &\leq (i-1)(n-1) + (2r+1-i+1)(2r-i-1) \\ &\quad + (r-2)(n+1-2r-2) \\ &= i^2 + i(n-4r-2) - (n-1) \\ &\quad + (2r-1)(2r+2) + (r-2)(n-2r-1). \end{aligned}$$

Since $n \geq 5r + 19$, it is easy to see that $i^2 + i(n-4r-2)$, consider as a

function of i , attains its maximum value when $i = r - 3$. Therefore,

$$\begin{aligned}\sigma(\pi) &\leq (r-3)^2 + (n-4r-2)(r-3) - (n-1) \\ &\quad + (2r-1)(2r+2) + (r-2)(n-2r-1) \\ &= (r-1)(2n-r) - 3(n-r) - n + 5r + 16 \\ &< \sigma(\pi),\end{aligned}$$

which is a contradiction.

Thus,

$$\sigma(K_{r+1} - Z, n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

for $n \geq 5r + 19$.

Acknowledgment

The authors thanks the referees for many helpful comments.

References

- [1] B. Bollobás, Extremal Graph Theory, Academic Press, London, 1978.
- [2] P. Erdős, On sequences of integers no one of which divides the product of two others and some related problems, Izv. Naustno-Issl. Mat. i Meh. Tomsk 2(1938), 74-82.
- [3] P. Erdős and T. Gallai, Graphs with given degrees of vertices, Math. Lapok, 11(1960), 264-274.
- [4] P. Erdős, M.S. Jacobson and J. Lehel, Graphs realizing the same degree sequences and their respective clique numbers, in Graph Theory, Combinatorics and Application, Vol. 1 (Y. Alavi et al., eds.), John Wiley and Sons, Inc., New York, 1991, 439-449.
- [5] R.J. Gould, M.S. Jacobson and J. Lehel, Potentially G -graphic degree sequences, in Combinatorics, Graph Theory and Algorithms, Vol. 2 (Y. Alavi et al., eds.), New Issues Press, Kalamazoo, MI, 1999, 451-460.
- [6] D.J. Kleitman and D.L. Wang, Algorithm for constructing graphs and digraphs with given valences and factors, Discrete Math., 6(1973), 79-88.

- [7] Chunhui Lai, A note on potentially $K_4 - e$ graphical sequences, Australasian J. of Combinatorics 24(2001), 123-127.
- [8] Chunhui Lai, An extremal problem on potentially $K_{p,1,1}$ -graphic sequences, Discrete Mathematics and Theoretical Computer Science 7(2005), 75-80.
- [9] Chunhui Lai, Potentially $K_{p,1,1,\dots,1}$ -graphic degree sequences, J. Zhangzhou Teachers College 17(4)(2004), 11-13.
- [10] Chunhui Lai, An extremal problem on potentially $K_m - C_4$ -graphic sequences, Journal of Combinatorial Mathematics and Combinatorial Computing, 61 (2007), 59-63.
- [11] Chunhui Lai, An extremal problem on potentially $K_m - P_k$ -graphic sequences, accepted by International Journal of Pure and Applied Mathematics.
- [12] Chunhui Lai and Lili Hu, An extremal problem on potentially $K_{r+1} - H$ -graphic sequences, accepted by Ars Combinatoria.
- [13] Chunhui Lai and Yuzhen Sun, An extremal problem on potentially $K_{r+1} - (kP_2 \cup tK_2)$ -graphic sequences, International Journal of Applied Mathematics & Statistics, 14(2009), 30-36.
- [14] Jiong-Sheng Li and Zi-Xia Song, An extremal problem on the potentially P_k -graphic sequences, Discrete Math., 212(2000), 223-231.
- [15] Jiong-Sheng Li and Zi-Xia Song, The smallest degree sum that yields potentially P_k -graphical sequences, J. Graph Theory, 29(1998), 63-72.
- [16] Jiong-sheng Li and Zi-Xia Song, On the potentially P_k -graphic sequences, Discrete Math. 195(1999), 255-262.
- [17] Jiong-sheng Li, Zi-Xia Song and Rong Luo, The Erdős-Jacobson-Lehel conjecture on potentially P_k -graphic sequence is true, Science in China(Series A), 41(5)(1998), 510-520.
- [18] Rong Luo, On potentially C_k -graphic sequences, Ars Combinatoria 64(2002), 301-318.
- [19] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48(1941), 436-452.
- [20] Jianhua Yin and Jiongsheng Li, Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size, Discrete Math., 301(2005) 218-227.

- [21] Jianhua Yin, Jiongsheng Li and Rui Mao, An extremal problem on the potentially $K_{r+1}-e$ -graphic sequences, *Ars Combinatoria*, 74(2005),151-159.
- [22] Mengxiao Yin, The smallest degree sum that yields potentially $K_{r+1}-K_3$ -graphic sequences, *Acta Math. Appl. Sin. Engl. Ser.* 22(2006), no. 3, 451-456.